Math 105 Chapter 10: Sequences and Series

Definition: that a sequence is an infinite ordered list of numbers.

$$\{a_n\} := \{a_1, a_2, \ldots, a_n, \ldots\}$$

eg $a_n = \frac{1}{n^2}$, $\{ \underset{\overset{\shortparallel}{a_1}}{1}, \underset{\overset{\shortparallel}{a_2}}{\frac{1}{4}}, \frac{1}{9}, \ldots \}$,

$a_n = (-1)^n$, $\{ \underset{\overset{\shortparallel}{a_1}}{-1}, \underset{\overset{\shortparallel}{a_2}}{1}, -1, 1, -1, \ldots \}$

$a_n = \frac{n}{n+1}$, $\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots \}$

Some times we don't know the formula that generates a sequence, but we can define them recursively, with initial conditions.

eg. $a_n = a_{n-1} + a_{n-2}$, $a_0 = 0$, $a_1 = 1$

So $a_2 = a_1 + a_0 = \quad + 1 = 1$

$a_3 = a_2 + a_1 = 1 + 1 = 2$

$a_4 = a_3 + a_2 = 2 + 1 = 3$

$\{a_n\} = \{0, 1, 1, 2, 3, 5, 8, 13, \ldots\}$

This sequence is know as the Fibonacci sequence.

Fun Fact: There is actually a closed form solution for the Fibonacci numbers that generates the sequence without a recurrence relation.

$$a_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$

eg $a_1 = 4$, $a_{n+1} = \frac{a_n}{a_n - 1} + n$, $n \geq 1$.

Lets see what the first few terms of the sequence are.

$$a_1 = 4$$

$$a_2 = \frac{a_1}{a_1 - 1} + 1 = \frac{4}{4-1} + 1 = \frac{7}{3}$$

$$a_3 = \frac{a_2}{a_2 - 1} + 2 = \frac{\frac{7}{3}}{\frac{7}{3} - 1} + 2 = \frac{15}{4}$$

$$a_4 = \frac{a_3}{a_3 - 1} + 3 = \frac{\frac{15}{4}}{\frac{15}{4} - 1} + 3 = \frac{48}{11}$$

Often times we are interested in learning about certain sequences and their properties.

eg $a_n = \frac{n}{n+1} = \left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \cdots \right\}$

What happens to a_n when n is very large?

Definition: Let $\{a_n\}$ be a sequence and $L \in \mathbb{R}$. We say that the $\{a_n\}$ converges to L if $\{a_n\}$ is sufficiently close to L for all n large enough. In this case we say:

"the limit as n approaches infinity of a_n is L"

or write:

$$\lim_{n\to\infty} a_n = L.$$

Otherwise we say $\{a_n\}$ is divergent.

Suppose $a_n = f(n)$ for some $f(x)$ then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} f(n)$$ from calc 1

eg $a_n = \frac{1}{n}$,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$$

So a_n converges to 0.

eg $a_n = \dfrac{3e^n}{1+e^n}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3e^n}{2+e^n} = \lim_{n\to\infty} \frac{3}{2e^{-n}+1} = \frac{3}{0+1} = 3$$

So a_n converges to 3

eg $a_n = n$

n gets arbitrarily large so

$$\lim_{n\to\infty} a_n = \infty$$

So a_n diverges to infinity.

eg $a_n = (-1)^n$

$$a_n = \{-1, 1, -1, 1, -1, 1, \dots\}$$

So a_n doesn't go to any number

$\lim_{n\to\infty} a_n$ does not exist.

So a_n diverges.

Properties of Limits

If $\lim_{n\to\infty} a_n$ exists and $\lim_{n\to\infty} b_n$ exists.

1. $\lim_{n\to\infty} (c a_n + b_n) = c \lim_{n\to\infty} a_n + \lim_{b\to\infty} b_n$ (linearity)

2. $\lim_{n\to\infty} a_n b_n = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right)$

3. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$, if $\lim_{n\to\infty} b_n \neq 0$

4. If $f(x)$ is a continuous function:

   $\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n)$

eg $\lim_{n\to\infty} \frac{3n^3+2n^2}{2n^3+3n+1} = \lim_{n\to\infty} \frac{3n^3+2n^2}{2n^3+3n+1} \frac{\frac{1}{n^3}}{\frac{1}{n^3}}$ ↖ divide by highest power trick

$$= \lim_{n\to\infty} \frac{3+\frac{2}{n}}{2+\frac{3}{n^2}+\frac{1}{n^3}}$$

$$= \frac{3+0}{2+0+0}$$

$$= \frac{3}{2}$$

Theorem: (Squeeze).

Suppose $b_n \le a_n \le c_n$ and for all $n \geq k$ for some k.

If $\lim_{n\to\infty} b_n = L = \lim_{n\to\infty} c_n = L$ then:

(1) $\lim_{n\to\infty} a_n$ exists

(2) $\lim_{n\to\infty} a_n = L$

eg Evaluate: $\lim_{n\to\infty} \frac{\sin(n)}{n}$

Since $-1 \le \sin(n) \le 1$

$$\Rightarrow \frac{-1}{n} \le \frac{\sin n}{n} \le \frac{1}{n}$$

$$\lim_{n\to\infty} \frac{-1}{n} = \lim_{n\to\infty} \frac{1}{n} = 0$$

So $\lim_{n\to\infty} \frac{\sin(n)}{n} = 0$

Definition: For $n = 0, 1, 2, 3, \ldots$ we define the factorial of n as

$$0! = 1$$
$$1! = 1$$
$$2! = 2 \cdot 1 = 2$$
$$3! = 3 \cdot 2 \cdot 1 = 6$$
$$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$$
$$\vdots$$
$$n! = n(n-1) \cdots 2 \cdot 1$$

In general $n! = n\,(n-1)!$

eg Find $\lim\limits_{n\to\infty} \frac{n!}{n^n}$. Let $a_n = \frac{n!}{n^n}$

$$a_n = \frac{1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-1) n}{n \cdot n \cdot n \cdot \cdots \quad n \cdot n}$$

$$= \frac{1}{n} \underbrace{\left(\frac{2}{n}\right)}_{\leq 1} \underbrace{\left(\frac{3}{n}\right)}_{\leq 1} \cdots \underbrace{\left(\frac{n-1}{n}\right)}_{\leq 1} \underbrace{\frac{n}{n}}_{\leq 1}$$

$$\leq \frac{1}{n}$$

So $0 \leq a_n \leq \frac{1}{n}$

Since $\lim\limits_{n\to\infty} \frac{1}{n} = 0$, $\lim\limits_{n\to\infty} a_n = 0$, by squeeze theorem.

Some definitions that will be useful.

<u>Definition:</u> Let a_n be a sequence. We say a_n is:

- <u>Bounded</u> if there is a $M \in \mathbb{R}$ such that

$$|a_n| \leq M, \text{ for all } n$$

- <u>Unbounded</u> if a_n is not bounded
- <u>(strictly) increasing</u> if

$$a_n \underset{(<)}{\leq} a_{n+1}, \text{ for all } n.$$

- <u>(strictly) decreasing</u> if

$$a_n \underset{(>)}{\geq} a_{n+1}, \text{ for all } n$$

- <u>monotone</u> if a_n is either increasing or decreasing.

eg $a_n = e^{-n}$. is bounded by 1 since

$$|a_n| \leq 1 \text{ for all } n.$$

. is strictly decreasing since

$$a_n > a_{n+1} \text{ for all } n.$$

eg $a_n = n^2$. is unbounded

. is strictly increasing.

<u>Theorem:</u> Bounded monotone sequences converge.

You will not be resposible for the above theorem, but is a useful fact to know.

Theorem $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} |a_n| = 0$

Geometric sequence: One important sequence that will come accross is

$$a_n = r^n \quad \text{for some } r \in \mathbb{R}.$$

What is $\lim_{n\to\infty} a_n$?

When $|r|<1$:

We have $0 \le |a_n| = |r|^n \to 0$ as $n\to\infty$

So $\lim_{n\to\infty} |a_n| = 0 \Rightarrow \lim_{n\to\infty} a_n = 0$

When $|r|>1$:

We have $|a_n| = |r|^n$ goes to infinity as n goes to infinity.

So a_n diverges.

When $r=1$

We have $a_n = 1$ so

$$\lim_{n\to\infty} a_n = 1$$

When $r=-1$

We have $a_n = (-1)^n$ which diverges.

To conclude: $$\lim_{n\to\infty} a_n = \begin{cases} 0 & , |r|<1 \\ 1 & , r=1 \\ \text{diverges} & , \text{otherwise.} \end{cases}$$

Infinite Series

Let $\{a_n\}$ be a sequences. One natural question one can ask is What happens if I add up all the terms? Eg.

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

Using sigma notation we can write the above expression as

$$\sum_{n=1}^{\infty} a_n \qquad (*)$$

For the remainder of the course we will discussing properties of $(*)$.

The first question we have to ask is what is $(*)$ even mean?

Let a_n be a sequence. We define the partial sum sequence by

$$\begin{aligned} S_n &= a_1 + a_2 + \cdots + a_n \\ &= \sum_{k=1}^{n} a_k \end{aligned}$$

Finally we define an infinite series by

$$\begin{aligned} \sum_{n=1}^{\infty} a_n &= \lim_{N\to\infty} S_N \\ &= \lim_{N\to\infty} \sum_{n=1}^{N} a_n \end{aligned}$$

Note! Compare the definition of $\sum_{n=1}^{\infty} a_n$ with

$$\int_1^{\infty} f(x)\,dx = \lim_{N\to\infty} \int_1^N f(x)\,dx$$

Definition: We say that $\sum_{n=1}^{\infty} a_n$ converges if S_n converges.
diverges if S_n diverges.

eg $\sum_{n=1}^{\infty} 1$

Note $a_n = 1$, so $S_N = \sum_{n=1}^{N} 1$

$$= N$$

So $\sum_{n=1}^{\infty} 1 = \lim_{N\to\infty} S_N = \lim_{N\to\infty} N = \infty$

So $\sum_{n=1}^{\infty} 1$ diverges to infinity.

eg $\sum_{n=0}^{\infty} (-1)^n$

So $a_n = (-1)^n$. We now need to find S_N.

$$S_0 = (-1)^0 = 1$$
$$S_1 = S_0 + (-1)^1 = 1 - 1 = 0$$
$$S_2 = S_1 + (-1)^2 = 0 + 1 = 1$$
$$S_3 = S_2 + (-1)^3 = 1 - 1 = 0$$
$$\vdots$$

So S_n alternate between 0 and 1 implying S_n diverges, thus.
$\sum_{n=0}^{\infty} (-1)^n$ diverges.

eg $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

We need to compute $S_N = \sum_{n=1}^{N} \frac{1}{n(n+1)}$

By partial fractions, we can observe

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

So
$$S_1 = 1 - \frac{1}{2}$$
$$S_2 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) = 1 - \frac{1}{3}$$
$$S_3 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) = 1 - \frac{1}{4}$$
$$\vdots$$
$$S_N = \left(1 \cancel{- \frac{1}{2}}\right) \cancel{+ \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots} \cancel{+ \left(\frac{1}{N-1} - \frac{1}{N}\right) + \left(\frac{1}{N}\right.} - \left.\frac{1}{N+1}\right)$$
$$= 1 - \frac{1}{N+1}$$

So $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{N \to \infty} S_N = \lim_{N \to \infty} 1 - \frac{1}{N+1} = 1$

This an example of a <u>telescoping series</u> (ie a series where successive terms cancel). <u>FUN</u>!

Geometric Series

One particularly nice series that comes up very often is a geometric series.

Definition: Let a, r ber real numbers, $a \neq 0$. a geometric series of the form

$$\sum_{n=0}^{\infty} ar^n$$

Note $a =$ first term, $r = \frac{2^{nd} \text{ term}}{1^{st} \text{ term}} = \frac{(n+1)^{th} \text{ term}}{n^{th} \text{ term}}$

We have
$$\sum_{n=0}^{\infty} ar^n = \begin{cases} \frac{a}{1-r}, & |r| < 1 \\ \text{diverges}, & |r| \geq 1 \end{cases}$$

Proof: When $r=1$; clearly $\sum_{n=0}^{\infty} ar^n = a \sum_{n=0}^{\infty} 1 = \infty$, so diverges.

So assume $r \neq 1$.

$$\begin{aligned} (1-r) S_N &= S_N - r S_N \\ &= \sum_{n=0}^{N} ar^n - r \sum_{n=0}^{N} ar^n \\ &= a + ar + ar^2 + \dots + ar^N \\ &\quad - (ar + ar^2 + \dots + ar^N + ar^{N+1}) \\ &= a - ar^{N+1} \end{aligned}$$

$$\Rightarrow \quad S_N = \frac{a(1-r^{N+1})}{1-r}$$

So $$\sum_{n=0}^{\infty} ar^n = \lim_{N\to\infty} S_N$$
$$= \lim_{N\to\infty} \frac{a(1-r^{N+1})}{1-r}$$
$$= \begin{cases} \frac{a}{1-r} & \text{if } |r|<1 \\ \text{diverges} & \text{if } r=-1, |r|>1 \end{cases}$$

Since $\lim_{N\to\infty} r^{N+1} = \begin{cases} 0 & \text{if } |r|<1 \\ \text{diverges} & \text{if } r=-1, |r|>1 \end{cases}$

Remark

· We have an important formula we derived
$$\sum_{n=0}^{N} ar^n = a\left(\frac{1-r^{N+1}}{1-r}\right)$$

eg Evaluate $\sum_{n=2}^{\infty} 2^{2n} 7^{1-n}$

$$\sum_{n=2}^{\infty} 2^{2n} 7^{1-n} = \sum_{n=2}^{\infty} (2^2)^n \frac{7}{7^n}$$
$$= \sum_{n=2}^{\infty} 7\left(\frac{4}{7}\right)^n$$

To use our formula we need to start at $n=0$, not 2. So let's reindex. Let $k=n-2$ so $n=k+2$

$$\sum_{n=2}^{\infty} 2^{2n} 7^{1-n} = \sum_{k+2=2}^{\infty} 7\left(\frac{4}{7}\right)^{k+2}$$
$$= \sum_{k=0}^{\infty} 7\left(\frac{4}{7}\right)^2\left(\frac{4}{7}\right)^k$$

Since $\left(\frac{4}{7}\right) < 1$ we have

$$\sum_{n=2}^{\infty} 2^{2n} 7^{1-n} = \frac{7\left(\frac{4}{7}\right)^2}{1-\frac{4}{7}}$$

$$= \frac{16}{3}$$

Eg Evaluate the gemetric series

$$-2 + \frac{5}{2} - \frac{25}{8} + \frac{125}{32} - \cdots$$

$a =$ first term $= -2$

$r = \dfrac{\text{second term}}{\text{first term}} = \dfrac{5/2}{-2} = \dfrac{-5}{4}$

so $-2 + \frac{5}{2} - \frac{25}{8} + \cdots = \sum_{n=0}^{\infty} -2\left(-\frac{5}{4}\right)^n$

since $|r| \geq 1$ we have the series diverges.

eg Write $2.15\overline{62} = 2.15626262\ldots$ as a ratio of integers.

$$2.15\overline{62} = 2.15 + \underbrace{\frac{62}{10^4} + \frac{62}{10^6} + \frac{62}{10^8} + \cdots}_{\text{geometric series}}$$

$a = \dfrac{62}{10^4}$,

$r = \dfrac{62/10^6}{62/10^4} = \dfrac{1}{10^2}$

So $\quad 2.15\overline{62} = 2.15 + \sum_{n=0}^{\infty} \frac{62}{10^4}\left(\frac{1}{10^2}\right)^n$

$$= 2.15 + \frac{62/10^4}{1 - \frac{1}{100}}$$

$$= 2.15 + \frac{62/10^4}{\frac{99}{100}}$$

$$= \frac{215}{100} + \frac{62}{9900}$$

$$= \frac{21347}{9900}$$

10.15